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ON PERIODS OF CUSP FORMS AND ALGEBRAIC CYCLES FOR $U(3)$

BY

S. GELBART*

Department of Mathematics
The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: mtgelbar@weizmann

AND

J. ROGAWSKI*

Department of Mathematics
UCLA, Los Angeles, California 90024, USA
e-mail: jonr@math.ucla.edu

AND

D. SOUDRY*

Department of Mathematics
Tel-Aviv University, Ramat Aviv 69978, Israel
e-mail: soudry@taurus.bitnet

ABSTRACT

In this paper we discuss relations between the following types of conditions on a representation π in a cuspidal L-packet of $U(3)$: (1) $L(s, \pi \times \xi)$ has a pole at $s = 1$ for some ξ; (2) a *period* of π over some algebraic cycle in $U(3)$ (coming from a unitary group in two variables) is non-zero; and (3) π is a *theta-series lifting* from some unitary group in two variables. As an application of our analysis, we show that the algebraic cycles on the $U(3)$ Shimura variety are *not* spanned (over the Hecke algebra) by the modular and Shimura curves coming from unitary subgroups.

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Introduction

The theory of automorphic forms and L-functions for the quasi-split unitary group

$$G = U(3)$$

has been developed by several authors in the last fifteen years. For a review of relevant results, methods, and unexplained notation, we refer the reader to [Ge].

In the present paper, we continue earlier investigations by analyzing the following types of conditions on a representation π in a cuspidal L-packet of G:

(1) the Langlands L-function $L(s, \pi \times \xi)$ has a pole at $s = 1$ for some ξ, i.e., π is "endoscopic" in the sense of [Ro1];

(2) a *period* of π over some algebraic cycle in $U(3)$ (coming from a unitary group in two variables) is non-zero; and

(3) π is a theta-series lifting from a unitary group $U(\Phi')$ in two variables.

In case π is *generic*, it turns out that all three of these conditions are equivalent (cf. §2.5 below). More precisely, take $U(\Phi')$ to be the quasi-split unitary group $U(1,1)$ over the quadratic extension E of F. Then the following are equivalent:

(1) $L(s, \pi \times \xi)$ has a pole at $s = 1$ for some fixed ξ ;

(2') for some φ in V_π, the *non-compact* period

$$\int_{U(1,1)(F)\backslash U(1,1)(\mathbb{A})} \varphi(h)\xi(\det h)dh$$

does not vanish; and

(3') an appropriate theta-series lift of π to $U(1,1)$ is not zero.

What is the situation for *arbitrary* (not necessarily generic) π? As remarked at the end of §8.9 of [GePS], the period in (2') is *always* zero if π is not generic; thus some substitute for this condition is needed. In the present paper, we make precise the notion of a generalized ("$U(2)$") *period* integral for any π (as appears in condition (2) above), and then we show that conditions (2) and (3') are equivalent. However, we also show that condition (1) — the existence of a pole for $L(s, \pi \times \xi)$ at $s = 1$ — is strictly *weaker* than both conditions (2) or (3').

More precisely, we construct cuspidal π such that the period integrals

$$\int_{G_c(F)\backslash G_c(\mathbb{A})} \varphi(r)\chi(\det r)dr$$

vanish identically for all characters χ of $U(1)$, and all cycles G_c in $U(3)$ coming from some $U(\Phi')$, even though $L(s, \pi \times \xi)$ has a pole at $s = 1$.

On the one hand, this result emphasizes an obvious point: whereas property (1) is shared by either *all* or *none* of the cuspidal members of a given L-packet, properties (2) and (3') can be shared by only *some*. Thus the real goal should be an *intrinsic* characterization of such properties, preferably in purely *local* terms (cf. §3.2, where, for example, a local criterion for (3') is conjectured).

On the other hand, this result is of special interest in the light of Tate's conjectures relating algebraic cycles to poles of L-functions (cf. [BlRo]). In particular, our methods imply that the algebraic cycles on the $U(3)$ Shimura variety are *not* spanned by the modular curves coming from "$U(2)$ subgroups"; see Section 2.9 for a precise statement and proof.

In Section 1, we collect several basic results about theta-liftings for the dual pairs $(U(\Phi'), U(3))$; among other things, we discuss the *irreducibility* of the theta-lifts, and we prove that the theta-lift to $U(3)$ of a cuspidal σ on $U(\Phi')$ is always non-zero for at least "most" choices of lifting data $(\psi, \gamma, \chi_1, \chi_2)$.

In Section 2 we define $U(2)$-periods, discuss the equivalence of conditions (2) and (3') for an arbitrary cuspidal π, and give an explicit formula for the (compact) periods of endoscopic hypercuspidal π. Using this formula, we also give examples (respectively) of hypercuspidal π which satisfy condition (1) but do *not* (respectively do) satisfy conditions (2) and (3'). We also explain the connection to algebraic cycles on Picard modular surfaces alluded to above.

It will be clear to the reader that these partial results leave largely untouched the problem of locally characterizing global properties of a general cuspidal π on $U(3)$. To underscore this fact, we close by describing some natural open problems concerning Howe lifts and theta-series correspondences.

We note that an apparently different approach to unitary periods was developed in [Kud]. We also wish to thank M. Harris for pointing out a mistake in an earlier version of this paper, S. Rallis for several helpful conversations, the referee for his comments, and Miriam Abraham for her patient typing (and retyping) of the manuscript.

(Minimal) Notation from [Ge]

(1) E is a quadratic extension of F.

(2) $U(3)$ denotes the unitary group of the three-dimensional skew-Hermitian

space (V,Φ) whose matrix with respect to the basis $\{\ell_{-1},\ell_0,\ell_1\}$ is

$$\Phi=\begin{pmatrix}0&0&1\\0&\xi&0\\-1&0&0\end{pmatrix}$$

with ξ in E satisfying $\bar{\xi}=-\xi$.

(3) The maximal unipotent subgroup N of $U(3)$ (preserving ℓ_{-1}) has the form

$$N=\{[w,t]\}=\left\{\begin{pmatrix}1&w\xi&z\\0&1&\bar{w}\\0&0&1\end{pmatrix}\right\}$$

with w in E and $z=\xi\frac{N(w)}{2}+t$, t in F.

(4) A cusp form φ on $U(3)$ is ***hypercuspidal*** if its Fourier coefficients along N are all zero; in particular, such a cusp form is not *generic*.

1. Basic Results on Theta-liftings

Throughout this Section, $U(\Phi')$ will denote the unitary group of a two-dimensional space Y with Hermitian form Φ'. When Y contains isotropic vectors, and we wish to emphasize this, we might use W in place of Y, and $U(1,1)$ in place of $U(\Phi')$; otherwise, when Y is anisotropic, we call $U(\Phi')$ a "compact $U(2)$".

1.1 Statement of results

For the dual reductive pair $(U(\Phi'),U(3))$, as for a general *unitary* dual pair, theta-series liftings are parametrized by lifting data $(\psi,\gamma,\chi_1,\chi_2)$, where ψ is a non-trivial character of $\mathbf{A}/F$, γ is a Hecke character of E whose restriction to $\mathbf{A}_F^x/F^x$ is $\omega_{E|F}$, and χ_1 and χ_2 are characters of $E^1\backslash E^1(\mathbf{A})$. Recall that:

(a) the data ψ,γ fixes a compatible embedding of $U(V\otimes Y)$ into the metaplectic group, hence a restriction of theta-functions Θ_Φ to $U(V\otimes Y)$ ([GeRo1], §3);

(b) the additional parameters χ_1, χ_2 serve to further fix the theta-functions on $U(V)\times U(\Phi')$ through the formula

$$\Theta_\Phi^{\chi_1,\chi_2}(g,h)=\chi_1(\det h)\chi_2(\det g)\Theta_\Phi(s(g,h))\ ;$$

and

(c) the data $(\psi, \gamma, \chi_1, \chi_2)$ determines a Howe lift and a theta-series lifting (cf. Chapter 2 of [Ge]); given a representation σ of $U(\Phi')$, we denote the corresponding theta-lift on $U(3)$ by $\Theta(\sigma, \psi)$ or $\Theta(\psi, \sigma)$ (suppressing the additional data γ, χ_1, χ_2 whenever possible).

In this Section, we collect some useful — partly folkloric — results concerning theta-series lifts for the dual pair $(U(\Phi'), U(3))$. These include the following assertions:

(a) The local Howe lift to $U(3)$ of an irreducible representation of a *compact* $U(2)$ is never generic.

(b) The theta-lift of a cuspidal π on $U(3)$ is *irreducible* on $U(\Phi')$; moreover
   (i) its lift back to $U(3)$ is π again; and
   (ii) the ψ-theta lift of any cuspidal σ on $U(1,1)$ is generic on $U(3)$ if and only if σ is ψ-generic (here "ψ-theta lift" refers to any theta-lifting with respect to the lifting data $(\psi, \gamma, \chi_1, \chi_2)$, ψ fixed).

(c) The ψ-theta-lift of any fixed cuspidal representation σ on a $U(\Phi')$ is non-zero on $U(3)$ for at least some choice of data (γ, χ_1, χ_2): more precisely, for any fixed ψ, γ *and* χ_2, we can find at least one χ_1 such that the $(\psi, \gamma, \chi_1, \chi_2)$ lift of σ is non-zero.

1.2 Irreducibility of theta-lifts ([Ra1])

The result we need is the same for a *general* reductive dual pair as for $(U(\Phi'), U(3))$. Hence we shall work now in this generality; in particular, as in Proposition 2.7 of [Ge], U_1, U_2 will denote any dual reductive pair, with pre-images $\tilde{U}_1, \tilde{U}_2$ in the metaplectic group.

PROPOSITION: *Suppose $\pi = \otimes \pi_v$ is a cuspidal representation of $\tilde{U}_1(\mathbf{A})$, with theta-series lifting $\Theta(\psi, \pi)$ to $\tilde{U}_2(\mathbf{A})$. Then:*

(a) *If $\Theta(\psi, \pi)$ is non-zero and cuspidal, it generates an irreducible representation $\sigma = \otimes \sigma_v$ of $\tilde{U}_2(\mathbf{A})$ with the property that each σ_v is the Howe lift $H(\psi, \pi_v)$ of π_v.*

(b) *The theta-lift of the cuspidal representation σ back to $\tilde{U}_1(\mathbf{A})$, denoted $\Theta(\psi, \sigma)$, is non-zero.*

(c) *If $\Theta(\psi, \sigma)$ is cuspidal, then it generates an irreducible cuspidal representation of $\tilde{U}_1(\mathbf{A})$ equivalent to π.*

Remarks: (1) Recall that $\Theta(\psi,\pi)$ does not denote a representation of $\tilde{U}_2(\mathbb{A})$, but rather a *space of functions* on $\tilde{U}_2(\mathbb{A})$, namely the theta-lifts

$$\int_{U_1(F)\backslash\tilde{U}_1(\mathbb{A})} \Theta_\Phi(g_1,g_2)\varphi(g_1)dg_1 = f_{\varphi,\Phi}(g_2)$$

as Φ runs through the space of ω_ψ and φ through the space V_π. Nevertheless, we shall continue to (sometimes) confuse an irreducible subspace of $L^2(U_i(F)\backslash\tilde{U}_i(\mathbb{A}))$ with the irreducible representation it generates; we hope this does not confuse the reader.

(2) Since we are dealing now with theta-lifts between *full inverse images* in the metaplectic group, only the parameter ψ is relevant. Indeed, the additional parameters (γ,χ_1,χ_2) are required only to fix an *embedding* of the groups U_i themselves (inside the metaplectic group). In particular, in the special context of $(U(3),U(\Phi'))$, we can fix embeddings of these groups in the appropriate metaplectic group and then apply Proposition 1.2 directly to cuspidal π and σ on the unitary groups themselves. Also, since "multiplicity one" is known for $U(3)=U_1$, we can conclude that the theta-lift of $\Theta(\psi,\pi)$ back to U_1 actually equals V_π, as soon as all the other assumptions of the Proposition are satisfied. We shall discuss this particular example further in §§1.4–1.9 below. ∎

1.3 Proof of Proposition 1.2 (assuming Howe's conjecture; cf. [Ra1])

(a) Since $\Theta(\psi,\pi)$ is assumed to be cuspidal (and non-zero), we may write

$$\Theta(\psi,\pi)=\bigoplus_i V_{\sigma_i}$$

with each V_{σ_i} an irreducible, non-empty subrepresentation of $L_0^2(U_2(F)\backslash\tilde{U}_2(\mathbb{A}))$. To prove that only one such V_{σ_i} can occur, we assume the contrary, i.e., that there are *two* distinct subspaces (at least). Then we can define the projection operator

$$\Theta(\psi,\pi)\xrightarrow{P} V_{\sigma_1}\oplus V_{\sigma_2}\ ,$$

and also an element T in $\mathrm{Bil}(\omega_\psi\times\pi,\ \sigma_1\oplus\sigma_2)\approx\mathrm{Hom}(\omega_\psi\otimes\pi,\ \sigma_1\oplus\sigma_2)$ by way of the formula

$$T(\Phi,\varphi)=P(f_{\varphi,\Phi})\ ,$$

for Φ in ω_ψ^∞, and φ in V_π. Note that

$$T(\omega_\psi(g_1,g_2)\Phi,\pi(g_1)\varphi)=\sigma_1\oplus\sigma_2(g_2)T(\Phi,\varphi)\ .$$

For any $\tilde{U}_2$ representation σ, let $\mathrm{Hom}_{\tilde{U}_1}^{\tilde{U}_2}(\omega_\psi \otimes \pi, \sigma)$ denote the space of L in $\mathrm{Hom}(\omega_\psi \otimes \pi, \sigma)$ such that

$$L(\omega_\psi(g_1, g_2)\Phi \otimes \pi(g_1)\varphi) = \sigma(g_2)L(\Phi, \varphi)$$

for all g_i in $\tilde{U}_i$. Then for each place v of F, and each $i = 1$ or 2, composition of the map T above with the natural projection onto σ_i produces a non-trivial element L_i of $\mathrm{Hom}_{\tilde{U}_1}^{\tilde{U}_2}(\omega_{\psi_v} \otimes \pi_v, \sigma_{i,v})$. In this way, using Howe's local duality conjecture, we shall derive a contradiction.

Given π_v as above, the Conjecture implies that $\mathrm{Hom}_{\tilde{U}_1 \times \tilde{U}_2}(\omega_{\psi_v}, \pi_v \otimes \sigma_v)$ (equivalently $\mathrm{Hom}_{\tilde{U}_1}^{\tilde{U}_2}(\omega_{\psi_v} \otimes \pi_v, \sigma_v)$) is non-zero only if σ_v is the uniquely determined Howe lift of π_v (and then this Hom space is one-dimensional). Applying this above, we conclude that our assumption (on the *reducibility* of $\Theta(\psi, \pi)$) yields (at least) two linearly independent elements of $\mathrm{Hom}_{\tilde{U}_1}^{\tilde{U}_2}(\omega_{\psi_v} \otimes \pi_v, \sigma_v)$ for all v; this is the desired contradiction to our assumption that $\Theta(\psi, \pi)$ is not irreducible.

(b) If $\sigma = \Theta(\psi, \pi)$ is cuspidal, then its theta-lift back to $\tilde{U}_1(\mathbb{A})$ is at least defined. Denote this lift by $\Theta(\psi, \sigma)$. To prove that $\Theta(\psi, \sigma) \neq 0$, we simply compute the inner product of any φ_f in $\Theta(\psi, \sigma)$, f in $\Theta(\psi, \pi)$, with an arbitrary φ in V_π. The result is

$$\begin{aligned}\int_{U_1(F)\backslash\tilde{U}_1(\mathbb{A})} \overline{\varphi(g)}\varphi_f(g)dg &= \int_{U_1(F)\backslash\tilde{U}_1(\mathbb{A})} \overline{\varphi(g)}\Big(\int_{U_2(F)\backslash\tilde{U}_2(\mathbb{A})} f(h)\overline{\Theta_\Phi(g,h)}dh\Big)dg \\ &= \int_{U_2(F)\backslash\tilde{U}_2(\mathbb{A})} f(h)\Big(\int \overline{\varphi(g)}\ \overline{\Theta_\Phi(g,h)}dg\Big)dh \\ &= \int_{U_2(F)\backslash\tilde{U}_2(\mathbb{A})} f(h)\overline{f_\varphi(h)}dh \neq 0\end{aligned}$$

since *both* f and f_φ are elements of the space of σ (and they are non-trivial for judicious choices of φ and f!).

(c) If $\Theta(\psi, \sigma)$ is assumed cuspidal, then by (b) and (a) together, we conclude $\Theta(\psi, \sigma)$ generates an irreducible representation π' of $\tilde{U}_1(\mathbb{A})$ *not* orthogonal to π. Thus π' must be equivalent to π.

Remarks (on the assumption of Howe's conjecture): To the best of our knowledge, there remain a few special cases of Howe's local conjecture which have not yet been proved in residue characteristic two (cf. [Wald] for the most general results available). Therefore, one must still exercise some care in applying Proposition 1.2 on the irreducibility of theta-lifts. In particular, we might *either* need to work with an irreducible *component* of this lifting; *or* assume that σ_v for "even" v is such that Howe's conjecture is already known to hold (e.g., U_v is compact, or σ_v is unramified). ∎

1.4 Specialization to $(U(3), U(\Phi'))$

In this setting, as we have already remarked, once splitting data is fixed, Proposition 1.2 applies directly to cuspidal representations π and σ of $U(3)$ and $U(\Phi')$; in particular, because multiplicity one holds for $U(3)$, we can conclude that the theta-lift of $\Theta(\pi)$ back to $U(3)$ actually equals π, *provided* all the other assumptions of Proposition 1.2 are satisfied.

So the question remains: in this case, are such cuspidality and non-vanishing assumptions automatically satisfied? The answer is generally "yes", as we shall now show in the sequence of results 1.5–1.8 below.

1.5 Towers of liftings

This method (which originated in [Ral]) gives a criterion for the cuspidality of a theta-lift. In the present context, it asserts that the lift of some π is cuspidal if and only if its lift to all suitable smaller unitary groups is zero.

1.5.1. PROPOSITION: *If π is a cuspidal automorphic representation of $U_1 = U(3)$, then its theta-lifts to $U_2 = U(\Phi')$ are cuspidal (though possibly zero).*

Proof: We may assume $U(\Phi') \approx U(1,1)$ (since otherwise there is nothing to prove). The Proposition is then a simple instance of the philosophy of "towers of liftings" described in §3.12 of [Ge], in this case for the tower

$$\begin{array}{ccl} & & \nearrow\ U(3,3) \\ & & \nearrow\ U(2,2) \\ U(3) & & \\ & & \searrow\ U(1,1) \\ & & \searrow\ U(\{0\}) \end{array}$$

Moreover, a direct proof results from the computations of §2.2 below, as we shall observe there. ∎

1.5.2 PROPOSITION: *Fix a cuspidal automorphic representation σ of $U(\Phi')(\mathbf{A})$, and lifting data $s = s(\psi, \gamma, \chi_1, \chi_2)$. Then the s-theta lift of σ to $U(3)$ is cuspidal if and only if the s-theta lift of $\sigma \otimes \gamma^1$ to $U(E)$ is zero. (Here γ^1 is the restriction of γ from E to the norm 1 elements of E.)*

N.B: This Proposition also amounts to a special case of the phenomenon of "towers of unitary liftings" described in §3.12 of [Ge]. ∎

Proof: Let φ be any function in the theta-lift of σ to $U(3)$. As expected, we must compute the constant term of φ in terms of the theta-lift of $\sigma\gamma^1$ to $U(E)$. In fact, we shall first compute the constant term of φ along the *center* U of the maximal unipotent subgroup N. To continue, we shall use a certain "mixed model" realization of the Weil representation of $\mathrm{Mp}(V \otimes Y)$ associated to the decomposition

$$V \otimes Y = (\ell_{-1}) \otimes Y + [(\ell_0) \otimes Y' \oplus (\ell_0) \otimes Y''] + (\ell_1) \otimes Y \ ,$$

where $Y = Y' \oplus Y''$ is any complete polarization of Y, and $\{\ell_{-1}, \ell_0, \ell_1\}$ is the usual basis for V. In this model, we recall that the Weil representation ω_ψ acts in the space of $\mathcal{S}((\ell_1) \otimes Y \otimes ((\ell_0) \otimes Y'') \approx \mathcal{S}(Y \times Y'')$, and the action of N is given by the formula

$$\text{(A.1.2)} \qquad \omega_\psi\left(\begin{pmatrix} 1 & \xi w & z \\ 0 & 1 & \bar{w} \\ 0 & 0 & 1 \end{pmatrix}, 1\right) \Phi(y, y'')$$

$$= \psi(2t(y,y) + 2tr(\xi w(y'', y_1)) + tr(\xi N(w)(y_{11}, y_1))\Phi(y, y'' + \bar{w} y_{11})$$

where

$$z = \xi\frac{N(w)}{2} + t,$$

with t in F. (Here $y = y_1 + y_{11}$ according to the decomposition $Y = Y' + Y''$.) Similarly, *if* Y is isotropic, *and* Φ' has the matrix form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $\{w_{-1}, w_1\}$, then the action of the subgroup

$$\left\{\begin{pmatrix} 1 & \xi s \\ 0 & 1 \end{pmatrix} : s \in F\right\}$$

stabilizing w_{-1} is given by the formula

$$\text{(A.1.4)} \qquad \omega_\psi\left(1, \begin{pmatrix} 1 & \xi s \\ 0 & 1 \end{pmatrix}\right) \Phi(y, y'') = \psi(2sN(\xi y''_*))\Phi(s \cdot y, y'') \ .$$

(Here $s \cdot y$ denotes the natural action of

$$\begin{pmatrix} 1 & \xi s \\ 0 & 1 \end{pmatrix}$$

on the vector y in Y, and $y'' = y''_* w_1$.) For a derivation of both formulas (A.1.2) and (A.1.4) above, see the Appendix to [Ge].

In particular, (A.1.2) implies that a typical element

$$\eta(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of U acts in $\mathcal{S}(Y, Y'')$ through the formula

$$\omega_\psi(s(\psi,\gamma)(\eta(t),h)\Phi(y,y'') = \psi(2(y,y)t)\Phi(y,y'') .$$

Thus we compute

$$\begin{aligned}\varphi_U(e) &= \int_{F\backslash \mathbb{A}} \int_{U(\Phi')\backslash U(\Phi')(\mathbb{A})} f(h)\Theta_\Phi(\eta(t),h)dh\ dt \\ &= \int \chi_1(\det h) f(h) \sum_{y,y''} \omega_\psi(s(1,h))\Phi(y,y'') \left(\int_{F\backslash \mathbb{A}} \psi((y,y)t)dt\right) dh \\ &= \int_{U(\Phi)\backslash U(\Phi')\mathbb{A}} \chi_1(\det h) f(h) \sum_{y''\in Y''} \sum_{(y,y)=0} \omega_\psi(s(1,h))\Phi(y,y'')dh .\end{aligned}$$

On the other hand, it is easy to check that if $\Phi(y,y'')$ is of the form $\Phi_1(y)\Phi_2(y'')$, then

(1.5.3)

$$\omega_\psi(s(\psi,\gamma)(1,h))\Phi(y,y'') = \omega^1_\psi(s_1(\psi,\gamma)(1,h))\Phi_1(y)\omega^2_\psi(s_2(\psi,\gamma)(1,h))\Phi_2(y'') .$$

Therefore,

$$\varphi_U(e) = \Phi_1(0) \int_{U(\Phi')\backslash U(\Phi')(\mathbb{A})} \gamma^1 \chi_1(\det h)(\sum_{y''\in Y''} \omega^2_\psi(s_2(\psi,\gamma)(1,h))\Phi_2(y'') f(h)dh$$

$$(1.5.4) \quad + \int_{U(\Phi')\backslash U(\Phi')(\mathbb{A})} \sum_{\substack{(Y,Y)=0 \\ y\neq 0}} \sum_{y''\in Y''} \omega_\psi(s(1,h))\Phi(y,y'')\ \chi_1(\det h) f(h)dh .$$

N.B: (1) In the first term on the right side of (1.5.3), $s_1(\psi,\gamma)$ refers to the embedding of $U(W^1\otimes Y)$ into $\mathrm{Mp}(W'\otimes Y)$ determined by the data (ψ,γ), where W' is the skew Hermitian space spanned by ℓ_{-1} and ℓ_1, and ω_ψ^1 is the corresponding Weil representation for the dual pair $(U(W'),U(Y))$. Similarly, $\omega_\psi^2\cdot s_2$ denotes the Weil representation for the pair $U((\ell_0))\times U(Y)$. (Note that h always denotes an element of $U(Y)$, but the *pair* $(1,h)$ has *three* different meanings in (1.6.1), being viewed alternately as an element of $U(V\otimes Y)$, $U(W'\otimes Y)$ or $U(E\otimes Y)$!)

(2) Since $U(Y)$ preserves the polarization $(\ell_{-1}\otimes Y)\oplus(\ell_1\otimes Y)$ of $W'\otimes Y$, it acts (essentially) linearly on Φ_1, viz.

$$\omega_\psi^1(s_1(1,h))\Phi_1(y)=\gamma^1(\det h)\Phi_1(hy)\ . \tag{1.5.5}$$

The γ^1 appearing in this formula is analogous to the one appearing in (3.4.2) of §3 of [Ge]. (More generally, *whenever* a complete polarization $Z\oplus Z^*$ comes into the picture — with or without the presence of a mixed model — and one member of the relevant unitary dual pair preserves this polarization, then its action through $\omega_\psi\cdot s$ is just the linear action twisted by $\gamma^1\cdot\det$.) ∎

Let us return now to the calculation of (φ_U and) φ_N. If Φ' is anisotropic, then the second term in (1.5.4) does not appear. On the other hand, if Φ' has the form

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

with respect to the isotropic vectors w_{-1} (generating Y') and w_1 (generating Y''), then the second term becomes

$$\int_{U_H(F)\backslash U(\Phi')(\mathbb{A})}\sum_{y''\in Y''}(\omega_\psi(s(1,h))\Phi)(w_{-1}y'')\chi_1(\det h)f(h)dh\ , \tag{1.5.6}$$

with $U_H\subset U(\Phi')$ the unipotent subgroup stabilizing W_1. (Here we are using (1.5.5), the fact that each non-zero y in Y is of the form $\gamma\cdot w_{-1}$ for some γ in $U(\Phi')(F)$ and the fact $\gamma'(\det h)=1$ for h in $U(\Phi')(F)$.) Our claim now is that integration over the full maximal unipotent subgroup N of $U(V)$ makes this second term above disappear altogether.

Indeed, applying formula (A.1.2) above, with $y=w_{-1}$ (i.e. $y_1=w_{-1}$ and $y_{11}=0$), implies that only the term in the integral with $y''=0$ survives the integration over N, i.e., the "second term" becomes

$$\int_{U_H(F)\backslash U(\Phi')(\mathbb{A})}\omega_\psi(s(1,h))\Phi(w_{-1},0)\chi_1(\det h)f(h)dh\ .$$

Then by formula (A.1.4) above, this last integral may be written

$$\int_{U_H(F)\backslash U(\Phi')(\mathbb{A})} \omega_\psi(s(1,h))\Phi(w_{-1},0)\chi_1(\det h)\int_{U_H(F)\backslash U_H(\mathbb{A})} f(sh)ds)dh\ ,$$

and this is identically zero since f was assumed to be cuspidal on $U(\Phi')$.

Returning to the first term in (1.5.4), we conclude

$$\begin{aligned}\varphi_N(1) &= \Phi_1(0)\int_{U(\Phi')\backslash U(\Phi')(\mathbb{A})} \gamma^1\chi_1(\det h)\sum \omega_\psi^2(s_2(1,h)\Phi_2(y'')f(h)dh\\ &= \Phi_1(0)\int_{U(\Phi')\backslash U(\Phi')(\mathbb{A})} \chi_1(\det h)\Theta_{\Phi_2}(s_2(1,h))\gamma^1(\det h)f(h)dh\end{aligned}$$

(since the first term in (1.5.4) is clearly invariant under $N(\mathbb{A})$). But this last integral clearly represents the s-theta-lift of $\sigma\otimes\gamma^1$ to $U(E)$. Thus we conclude φ is cuspidal if and only if this lift vanishes. ∎

1.6 L-functions and non-vanishing of theta-lifts

In this paragraph we give a criterion for the theta-lift of π from $U(3)$ to $U(\Phi')$ to be non-zero, and show that the non-vanishing of this lift already implies that π itself is the theta-lift of some cuspidal σ on $U(\Phi')$, i.e. every φ in π is a theta-integral lift of some f in σ (even if we can't prove yet that the full theta-lift of σ is irreducible).

PROPOSITION:

(a) *If $L(s,\pi\times\xi)$ has a pole at $s=1$ for some ξ, then the theta-lift of π to $U(\Phi')$ is non-trivial (for some $U(\Phi')$ and lifting data $(\psi,\gamma,\chi_1,\chi_2)$);*

(b) *Suppose the theta-lift of π to $U(\Phi')$ is non-zero (for the lifting data ψ,γ,χ_1, χ_2) and σ is an irreducible component of this lift. Then its theta-lift back to $U(3)$ will be cuspidal, and equal to V_π. More precisely, for any b in F^x, the $(\psi_b,\gamma,\chi_1,\chi_2)$ lift of $\sigma\otimes\gamma^1$ to $U(E)$ will be zero, so by Proposition 1.5.2 the $(\psi_b,\gamma,\chi_1,\chi_2)$ theta-lift of σ to $U(3)$ will be cuspidal (for any b); moreover, for $b=1$, the theta-lift of σ back to $U(3)$ contains π.*

Proof: Although (a) and (b) are discussed in the proof of Theorem 6.1.1 of [GeRo1], we fill in the details of the proof of (b) since they are lacking there. By Proposition 1.5.1, we know that σ is cuspidal automorphic. So for any b in F^x, let π_b denote the theta-lift of σ to $U(3)$ corresponding to the data $s_b=(\psi_b,\gamma,\chi_1,\chi_2)$. By Proposition 1.5.2 and multiplicity one for $U(3)$, we know that π_b (with $b=1$)

will contain π as soon as we know this π_b is cuspidal. But by Proposition 1.5.2, π_b is cuspidal (for any fixed b) if and only if the s_b-lift of $\sigma \otimes \gamma^1$ to $U(E)$ is zero. So let us assume this is *not* so, and then derive from this assumption a contradiction.

Indeed, assuming $\varphi_N \not\equiv 0$ for φ in the space of π_b, we construct a map

$$\pi_b \longrightarrow \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$$

(for appropriate χ) by integrating the constant terms for π_b against the s-lift of $\sigma \otimes \gamma^1$ to $U(1)$. More precisely, let μ be a character of $U(E)\backslash U(E)(\mathbf{A})$ such that

(1.6.1)
$$\int_{U(1)\backslash U(1)(\mathbb{A})} \int_{U(\Phi')\backslash U(\Phi')(\mathbb{A})} \Theta_{\Phi_2}(s_2(\varepsilon, h))\gamma^1(\det h) f(h)\mu^{-1}\gamma^1(\varepsilon) dh d\varepsilon \neq 0 \ ,$$

and define the functional T on π_b by

$$T(\varphi) = \int_{U(1)\backslash U(1)(\mathbb{A})} \varphi_N \begin{pmatrix} 1 & & \\ & \varepsilon & \\ & & 1 \end{pmatrix} \mu^{-1}\gamma^1(\varepsilon) d\varepsilon \ . \tag{1.6.2}$$

Note that by the formula for φ_N derived at the end of the proof of Proposition 1.5.2, it is clear that

$$T(\varphi) = \text{the expression (1.6.1) above} \ .$$

Thus it follows that $T \not\equiv 0$.

On the other hand, it is obvious from (1.6.2) that

$$T(\pi_b(n)\varphi) = T(\varphi)$$

and

$$T(\pi_b \begin{pmatrix} 1 & & \\ & \varepsilon & \\ & & 1 \end{pmatrix} \varphi) = \mu(\gamma^1)^{-1}(\varepsilon) T(\varphi) \ ,$$

for n in $N(\mathbf{A})$ and $|\varepsilon| = 1$.

Now set

$$\chi \begin{pmatrix} \alpha & * & * \\ 0 & \varepsilon & * \\ 0 & 0 & \bar{\alpha}^{-1} \end{pmatrix} = \mu^{-1}\left(\frac{\alpha}{\bar{\alpha}}\right) \mu\gamma^{-1}\left(\frac{\alpha}{\bar{\alpha}}\, \varepsilon\right) \ .$$

To prove our claim that T intertwines π_b with ind χ, it remains to check that

$$T(\pi_b \begin{pmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{pmatrix} \varphi) = |\alpha|\gamma(\bar{\alpha})^2 T(\varphi) .$$

For this, just observe that

$$\begin{pmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{pmatrix} \quad \text{takes} \quad (y, y'') = \ell_1 \otimes y + \ell_0 \otimes y'' \quad \text{to} \quad (\bar{\alpha} y, y'')$$

(which action has determinant $\bar{\alpha}^2$). Thus

$$\omega_\psi(s \begin{pmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{pmatrix}, h)\varphi(0, y'') = |\alpha|\gamma(\bar{\alpha})^2 \omega_\psi(s(1, h))\varphi(0, y'') ,$$

and the desired identity follows.

To obtain the desired contradiction, note that (any irreducible constituent of) π_b is almost everywhere equivalent to π. Indeed, at almost every place v (namely where b is a norm), the lift Howe $(\psi_b, \gamma, \chi_1, \chi_2)(\sigma_v)$ is independent of b, and hence π_v is equivalent to $(\pi_b)_v$. This implies that π itself is a CAP representation in the sense of [PS], i.e., it is almost everywhere equivalent to a constituent of some $\mathrm{Ind}_{B(\mathbf{A})}^{G(\mathbf{A})} \chi$. On the other hand, the results of [Rol] imply that all CAP representation of $G(\mathbf{A})$ must lie in some A-packet $\prod(\rho')$ with ρ' a unitary character of $U(1,1) \times U(1)$ (cf. Chapter 6 of [Ge]).

More precisely, the exact form of χ described above implies that each constituent of Ind χ belongs to a packet of the form $\prod(\rho)$ with $\rho = \rho(\theta)$ and θ semi-regular; cf. pages 173–174 of [Rol]. In particular, by strong multiplicity one for L-packets, π itself belongs to such a packet, and this contradicts Theorem 13.2.2 of [Rol]. ∎

1.7 **Proposition** (Whittaker models and non-vanishing of theta-lifts): *Suppose $U(\Phi') = U(1,1)$, and σ is any cuspidal representation of $U(\Phi')$ (not necessarily of the form $\Theta(\pi)$ as in Proposition 1.2). For any fixed additive character ψ of $F\backslash\mathbf{A}$, the $(\psi, \gamma, \chi_1, \chi_2)$-theta-lift of σ to $U(3)$ will automatically be non-zero (for any γ, χ_1 and χ_2) provided σ is ψ-generic, i.e., its space of ψ-th Fourier coefficients*

$$\mathcal{W}(\sigma, \psi) = \left\{ \int_{U_H \backslash U_H(\mathbf{A})} f(sh)\psi(s)ds : f \in V_\sigma \right\}$$

is non-zero.

Proof: Let π denote the ψ-theta-lift of σ to $U(3)$ (with respect to the data $(\psi, \gamma, \chi_1, \chi_2)$), and pick φ arbitrary in π. To show that $\pi \neq \{0\}$, we shall compute the "ψ-Fourier coefficient" W_φ^ψ of φ directly in terms of $\mathcal{W}(\sigma, \psi)$ (and hence show that $W_\varphi^\psi \neq 0$ for at least some φ).

Recall that

$$W_\varphi^\psi(1) = \int_{N(F)\backslash N(\mathbf{A})} \varphi(n)\overline{\psi_N(n)}dn = \int_{N(F)U(\mathbf{A})\backslash N(\mathbf{A})} \varphi_U(n)\overline{\psi_N(n)}dn$$

where

$$\psi_N(n) = \psi_N \begin{pmatrix} 1 & \xi w & z \\ 0 & 1 & \bar{w} \\ 0 & 0 & 1 \end{pmatrix} = \psi(\operatorname{Im} w) ,$$

and φ_U is the constant term of φ along.

$$U = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

already computed in the proof of Proposition 1.6. Starting from formula (1.6.2) for φ_U, we again integrate over $N(F)U(\mathbf{A})\backslash N(\mathbf{A})$ using formula (A.1.2) of that Section. Since the first term in (1.5.4) was invariant under $N(\mathbf{A})$, the presence of the non-trivial character $\psi_N(n)$ in the integration over $N(\mathbf{A})$ now implies that this first term makes *no* contribution to $W_\varphi^\psi(1)$. As for the second term, it follows from expression (1.5.6), together with (A.1.2) applied with $y = w_{-1}$, that (this time) only the term with

$$y'' = \frac{1}{4\xi^2}\, w_1 = w_1^*$$

survives the integration against $\psi_N(n)$. Hence we get

$$W_\varphi^\psi(1) = \int_{U_H(F)\backslash U(1,1)(\mathbf{A})} \omega_\psi(s(1,h))\Phi(w_{-1}, w_1^*)\chi_1(\det h)f(h)dh .$$

Integrating now over $U_H(F)\backslash U_H(\mathbf{A})$ using formula (A.1.4) gives us (finally)

$$W_\varphi^\psi(1) =$$
$$\int_{U_H(\mathbf{A})\backslash U(1,1)(\mathbf{A})} \omega_\psi(s(1,h))\Phi(w_{-1}, w_1^*)(\int_{U_H\backslash U_H(\mathbf{A})} f(sh)\psi(sN(\frac{1}{\xi})ds)\chi_1(h)dh ,$$

i.e.,

$$W_\varphi^\psi(1) = \int_{U_H(\mathbb{A})\backslash U(1,1)(\mathbb{A})} \omega_\psi(s(1,h))\Phi(w_{-1}, w_1^*) W_f^{\psi^{N(\frac{1}{\xi})}}(h)\chi_1(\det h)dh \,.$$

Note that $W_f^{\psi^{N(\frac{1}{\xi})}}$ is in the B_H-orbit of

$$W_f^\psi(h) = \int f(sh)\psi(s)ds \;;$$

indeed

$$W_f^{\psi^{N(\xi^{-1})}}(h) = W_f^\psi\left(\begin{pmatrix} \xi^{-1} & 0 \\ 0 & 1 \end{pmatrix} h\right) \,.$$

Thus $\mathcal{W}(\sigma,\psi) \neq \{0\}$ implies $\mathcal{W}(\sigma,\psi^{N(\xi^{-1})}) \neq \{0\}$, and the latter implies $W_\varphi^\psi(1) \neq 0$ (for appropriately chosen Φ). Therefore we have not only proved the Proposition, but also the (stronger):

COROLLARY 1.7.1: *For any f in V_σ, with σ cuspidal on $U(1,1)$, let φ denote the $(\psi,\gamma,\chi_1,\chi_2)$ theta-lift of f to $U(3)$. Then*
(1.7.2)

$$W_\varphi^\psi(1) = \int_{U_H(\mathbb{A})\backslash U(1,1)(\mathbb{A})} \omega_\psi(s(1,h))\Phi(w_{-1}, w_1^*)\chi_1(\det h) W_f^{\psi^{N(\xi^{-1})}}(h)dh \,.$$

In particular, $\mathcal{W}(\pi,\psi) \neq \{0\}$ (which means π is generic) if and only if $\mathcal{W}(\sigma,\psi) \neq \{0\}$ (i.e. σ is ψ-generic). ∎

Concluding Remark: From the computations of this Section, it is clear that theta-lift to $U(3)$ from a *compact* $U(\Phi')$ can *never* be generic. Indeed, in this case the second term in (1.5.4) does not appear (and the invariance of the first under N implies that integration against any non-trivial character ψ_N gives zero). As we shall see below, this fact also follows from its local analogue (Proposition 1.9). ∎

1.8 General non-vanishing of theta-lifts

In this paragraph, Y is *any* Hermitian space of dimension 2 over E, and σ *any* cuspidal representation of $U(Y)(\mathbb{A})$.

PROPOSITION: *For any fixed character of ψ of $\mathbf{A}/F$, we can choose data (γ, χ_1, χ_2) such that the space of theta-series liftings $\Theta_{\psi,\gamma}^{\chi_1,\chi_2}(\sigma)$ is non-zero on $U(V)(\mathbf{A}) = U(3)(\mathbf{A})$.*

Proof: Take f in V_σ and consider φ_f in $\Theta_{\psi,\gamma}^{\chi_1,\chi_2}(\sigma)$ defined by the formula

$$\varphi_f(g) = \int_{U(Y)\backslash U(Y)(\mathbf{A})} \Theta_\Phi^{\chi_1,\chi_2}(s(\psi,\gamma)(g,h))f(h)dh .$$

To show that φ is non-zero for some choice of f, Φ and data γ, χ_1, χ_2, we shall examine the formula for the c-th Fourier coefficient of φ along the unipotent subgroup

$$U = \{\eta(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in F\}$$

with $c \neq 0$. For this, it is convenient to use the same mixed-model realization of ω_ψ as used in §1.5.

By definition,

$$\begin{aligned}\varphi_{\psi_c}(1) &= \int_{F\backslash\mathbf{A}} \varphi_f(\eta(t))\overline{\psi(ct)}dt \\ &= \int_{F\backslash\mathbf{A}} \left(\int_{U(Y)\backslash U(Y)(\mathbf{A})} f(h)\Theta_\Phi^{\chi_1,\chi_2}(\eta(t),h)dh \right) \overline{\psi(ct)}dt .\end{aligned}$$

Realizing ω_ψ in $\mathcal{S}(Y, Y'')$ as in §(1.5), we have

$$\Theta_\Phi^{\chi_1,\chi_2}(\eta(t),h) = \chi_1(\det h) \sum_{y\in Y} \sum_{y''\in Y''} \omega_\psi(\eta(t),h)\Phi(y,y'') .$$

Therefore, using formula (A.1.2) of that paragraph (with $w = 0$) we have

$$\varphi_{\psi_c}(1) = \int_{U(Y)\backslash U(Y)(\mathbf{A})} \sum_{(y,y)=\frac{c}{2}} \sum_{y''} \omega_\psi(1,h)\Phi(y,y'')\chi_1(\det h)f(h)dh .$$

Using formula (1.5.3) then gives

$$\varphi_{\psi_c}(1) = \int_{U(Y)\backslash U(Y)(\mathbf{A})} \chi_1(\det h)f(h) \left(\sum_{y''\in Y''} \omega_\psi^2(1,h)\Phi_2(y'') \right) \times \left(\sum_{U_c(Y)\backslash U(Y)(F)} \omega_\psi^1(1,h)\Phi_1(\gamma w_c) \right) dh$$

where w_c is a fixed vector in Y of length $\frac{c}{2}$, and $U_c(Y)$ is its isotropy group in $U(Y)$.

Now note that the sum

$$\sum_{y'' \in Y''} \omega_\psi^2(s_2(1,h)\Phi_2(y'')$$

just represents the theta-kernel $\Theta_{\Phi_2}(1,h)$ associated to the dual pair $(U(E), U(Y))$ (with lifting data $(\psi, \gamma, \chi_1, \chi_2)$). Recall also (cf. (1.5.5)) that

$$\omega_\psi^1(s_1(1,h)\Phi_1(\gamma w_c) = w_\psi^1(s_1(1,\gamma h)\Phi_1(w_c) \ .$$

Thus we may write

$$\begin{aligned}\varphi_{\psi_c}(e) &= \int_{U_c(Y)(F)\backslash U(Y)(\mathbb{A})} \chi_1\gamma^1(\det h)\Phi_1(hw_c)\Theta_{\Phi_2}(1,h)f(h)dh \\ &= \int_{U_c(Y)(\mathbb{A})\backslash U(Y)(\mathbb{A})} \chi_1\gamma^1(\det h)\Phi_1(w_c h) \\ &\qquad \left(\int_{U_c(Y)(F)\backslash U_c(Y)(\mathbb{A})} \chi_1\chi^1(\det h)\Theta_{\Phi_2}(1,rh)f(rh)dr\right) dh \ .\end{aligned}$$

Note now that $U_c(Y)\backslash U_c(Y)(\mathbb{A})$ is a compact abelian group isomorphic to $E^1\backslash E^1(\mathbf{A})$, and for a suitable choice of Φ_2 and h_0,

$$r \longrightarrow \Theta_{\Phi_2}(rh_0)f(rh_0)$$

is a *non-zero* continuous function on this group. On the other hand, $\chi_1\gamma^1(\det h)\Phi_1(w_c h)$ represents a fairly general function on $U_c(Y)(\mathbb{A})\backslash U(Y)(\mathbb{A})$, since Φ_1 is arbitrary. Therefore, picking $\chi_1\gamma^1$ so that the $\chi_1\gamma^1$-th Fourier coefficient of $\Theta_{\Phi_2}(rh_0)f(rh_0)$ on $U_c(Y)(F)\backslash U_c(Y)(\mathbb{A})$ is non-zero, we conclude that the inner integral above, and hence the double integral for a suitable choice of Φ_1, will be non-zero. ∎

Remarks: (1) Although this result is useful (and indeed will be used later in this paper), it has many shortcomings: the most obvious is that we don't know *which* data (γ, χ_1, χ_2) will make $\pi = \Theta_{\psi,\gamma}^{\chi_1\chi_2}(\sigma)$ non-zero; hence we can't be sure which L-packet $\prod(\rho)$ on $U(3)$ contains π. We shall return to this question in §3; a more desireable result would be a criterion for checking directly whether a given lift $\Theta_{\psi,\gamma}^{\chi_1\chi_2}(\sigma)$ is zero or not.

(2) We may reformulate this Proposition as saying that for fixed ψ *and* γ, there exists χ_1 such that

$$\Theta_{\psi,\gamma}^{\chi_1\chi_2}(\sigma) \neq 0 \ .$$

Indeed, at the crucial endpoint of the proof, we just need to choose χ_1 such that the $\chi_1\gamma^1$-th Fourier coefficient of $\Theta_{\Phi_2}(rh_0)f(rh_0)$ is non-zero. Thus the Proposition really asserts that for given data ψ, γ and χ_2, there exists χ_1 such that

$$\Theta_{\psi,\gamma}^{1,\chi_2}(\sigma \otimes \chi_1) \neq \{0\} \ . \qquad \blacksquare$$

1.9 Local Howe lifts from a compact $U(2)$

Suppose Y is an *anisotropic* Hermitian space of dimension two over the *local* quadratic extension E of F, and σ is an irreducible admissible representation of $U(Y)$. Because Howe's conjecture holds for arbitrary representations of the *compact* group $U(Y)$, the Howe lift of σ to $U(3)$ can be defined as in Section 2.4 and 3.11 of [Ge]; like a global theta-series lifting, it depends not only on the choice of an additive character ψ, but also on the additional lifting data γ, χ_1 and χ_2.

PROPOSITION: *The Howe lift of σ from $U(Y)$ to $G = U(3)$ is degenerate, i.e., it supports no non-trivial Whittaker functionals.*

Proof: Without loss of generality, we may assume $\chi_1 = \chi_2 \equiv 1$. Then we let π denote the Howe lift of σ to G relative to the splitting data $s = s(\psi,\gamma)$. Because $U(Y)$ is compact, π simply acts in the σ-isotypic component of the space of the metaplectic representation ω_ψ relative to the action of $\omega_\psi \circ s$ restricted to $U(V) \times U(Y)$. Let V_π denote this isotypic subspace.

By a Whittaker functional for π, we mean a linear functional $\ell : V_\pi \longrightarrow \mathbb{C}$ satisfying

$$\ell(\omega_\psi \circ s(n)v) = \ell(\pi(n)v) = \psi_N(n)\ell(v) \tag{1.9.1}$$

for all v in V_π, n in the standard maximal unipotent subgroup N of G, and ψ_N a non-trivial character of N modulo its center

$$U = \left\{ \eta(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} .$$

We must prove that any such ℓ is 0.

By our assumption that π is the Howe lift of σ, we have (cf. §1.3) a non-zero map T from $\omega_\psi \circ s(\psi,\gamma)\otimes\sigma$ to π satisfying

$$T(\omega_\psi(g,h)\Phi\otimes\sigma(h)f)=\pi(g)T(\Phi\otimes f)\ .$$

So composing T with any Whittaker functional ℓ on π define a bilinear form $\langle\Phi,f\rangle$ on $\omega_\psi\times\sigma$ such that

$$\langle\omega_\psi\begin{pmatrix}1 & w\xi & \frac{1}{2}w\bar{w}\xi+t\\ 0 & 1 & \bar{w}\\ 0 & 0 & 1\end{pmatrix}\varphi,f\rangle=\psi_N(n)\langle\varphi,f\rangle\ . \tag{1.9.2}$$

But T must be surjective (by the irreducibility of π). Thus it suffices to show that $<\Phi,f>\equiv 0$.

For this, we (again) used the "mixed model" for ω_ψ associated to the decomposition

$$V\otimes Y=(\ell_{-1})\otimes Y+[(\ell_0)\otimes Y'\oplus\ell_0\otimes Y'']+(\ell_1\otimes Y$$

where $Y=Y'\oplus Y''$ is any polarization of Y and $\{\ell_{-1},\ell_0,\ell_1\}$ is the usual basis for V. In this model, ω_ψ acts in the space $\mathcal{S}(\ell_1\otimes Y\oplus\ell_0\otimes Y'')\approx\mathcal{S}(Y\otimes Y'')$, and the action of N is given by the following formula

$$\omega_\psi\left(\begin{pmatrix}1 & \xi w & z\\ 0 & 1 & \bar{w}\\ 0 & 0 & 1\end{pmatrix},1\right)\Phi(y,y'') \tag{1.9.3}$$

$$=\psi(2t(y,y)+2tr(\xi w(y'',y_1))+tr(\xi N(w)(y_{11},y_1))\Phi(y,y''+\bar{w}y_{11})$$

where

$$z=\xi\ \frac{N(w)}{2}\ t\quad(\text{with } t \text{ in } F).$$

(Here $y=y_1+y_{11}$ according to the decomposition $Y'+Y''$, as in the global analogue used in paragraphs 1.5 and 1.7.) In particular, for $\eta(t)$ in U, we have

$$\omega_\psi(\eta(t),1)\Phi=(\psi_t^*\Phi_1)\otimes\Phi_2$$

for $\Phi=\Phi_1\otimes\Phi_2$ and $\psi_t^*\Phi_1(y)=\psi(2t(y,y))\Phi_1(y)$. Therefore, from (1.9.2), it follows that for each fixed Φ_2 and f, the distribution

$$D:\Phi_1\longrightarrow\langle\Phi_1\otimes\Phi_2,f\rangle$$

satisfies

$$D(\psi_t^*\Phi_1) = D(\Phi_1)$$

for all t in F. Thus, as is well-known, D must be supported at the origin.

Let us first suppose F is non-archimedean. In this case, $\Phi_1 \longrightarrow D(\Phi_1)$ supported at the origin implies that

$$\langle \Phi_1 \otimes \Phi_2, f\rangle = \alpha(\Phi_2, f)\Phi_1(0)$$

for some bilinear form α on $\mathcal{S}(Y') \times V_\sigma$. On the other hand, applying (1.9.3) (with $y = 0$) says $\omega_\psi(n, 1)\Phi(0, y'') = \Phi(0, y'')$. Thus (1.9.2) implies

$$\begin{aligned}\psi_N(n)\alpha(\Phi_2, f)\Phi_1(0) &= \psi_N(n)\langle \Phi_1 \otimes \Phi_2, f\rangle \\ &= \langle \omega_\psi(n)\Phi, f\rangle \\ &= \alpha(\Phi_2, f)\Phi_1(0)\end{aligned}$$

for all n in $U\backslash N$ and Φ_1 in $\mathcal{S}(Y)$. Thus we conclude $\alpha(\Phi_2, f) \equiv 0$, i.e., $< \varphi, f >= 0$, and the Whittaker functional ℓ is zero as claimed.

In the archimedean case, the proof is similar but a bit more complicated. In this case, $\Phi_1 \longrightarrow \langle \Phi_1 \otimes \Phi_2, f\rangle$ defines a distribution on $C_c^\infty(Y)$ instead of $\mathcal{S}(Y)$, but since the trilinear form $\langle \Phi_1 \otimes \Phi_2, f\rangle$ is determined by its pull-back to $C_c^\infty(Y) \times \mathcal{S}(Y') \times V_\sigma$, it still suffices to show that D is always zero. More significantly, the fact that D is supported at 0 now implies only that

$$\langle \Phi_1 \otimes \Phi_2, f \langle = \sum_I \alpha_I(\Phi_2, f) D^I(\Phi_1)(0) ,$$

where D^I is the differential operator $\frac{\partial^{i_1}}{\partial v'^{i_1}} \frac{\partial^{i_2}}{\partial v''^{i_2}}$ (and $y = v' + v''$ according to the decomposition $Y = Y' + Y''$). So this time we have to apply D^I to the right hand side of (1.9.3) *before* setting $y = 0$. The result is that (1.9.2) now implies the contradiction that $\psi_N(n)$ is a *polynomial* in the variables of n (instead of just 1), unless *all* the $(\alpha_I(\Phi_2, f) = 0$ (respectively just $\alpha(\Phi_2, f) = 0$ as in the non-archimedean case). Thus we still conclude that $< \Phi_1 \otimes \Phi_2, f >\equiv 0$, i.e., $\ell = 0$. ∎

Remark: Proposition 1.9 works independently of the choice of ψ_N, ψ or γ. ∎

2. Analysis of $U(2)$-Periods

2.1 Definitions

As usual, let G denote the standard quasi-split unitary group $U(V) = U(3)$, and let π be a fixed cuspidal automorphic representation of $G(\mathbb{A})$. The space of *periods* of π is determined by a certain $U(2)$-subgroup H_c of G and an automorphic character χ of $E^1(\mathbb{A})$.

More precisely, for any c in F^x, let Φ_c denote a Hermitian form in two variables such that $\det(\Phi_c) = -c$, and let $H_c = U(\Phi_c)$. Note that:

(i) the isomorphism class of H_c depends only on the class of c modulo $N_{E/F}(E^x)$;

(ii) H_c is quasi-split (and isomorphic to $U(1,1)$) if and only if $c \in N_{E/F}(E^x)$; and

(iii) the group H_c embeds in G, uniquely up to conjugation, as the stabilizer G_c in G of a vector v_c of "length" c.

N.B: Since the form on V is *skew*-Hermitian, v of "length" c means $(v, v) = c\xi$. ∎

Definition: For any φ in V_π, c in F^x, and character χ of $E^1(\mathbb{A})/E^1$, define the **period** $P(\varphi, c, \chi)$ by the convergent integral

$$P(\varphi, c, \chi) = \int_{G_c(F)\backslash G_c(\mathbb{A})} \varphi(r)\chi(\det r)dr .$$

Then set

$$P(\pi, c, \chi) = \{P(\varphi, c, \chi) : \varphi \in V_\pi\} .$$

This is the (c, χ) period of the (cuspidal) representation π. ∎

2.2 The relation between $P(\pi, c, \chi)$ and theta-liftings

We fix $U(1,1)$ to act on the *Hermitian* space $W = Ew_1 \oplus Ew_2$, with corresponding Hermitian form

$$\Phi' = \begin{pmatrix} 0 & \xi^{-1} \\ -\xi^{-1} & 0 \end{pmatrix} .$$

Since $\det(\Phi') = -1$ modulo norms, $U(W) \approx U(1,1) \approx H_1$. Its derived group is simply $\mathrm{SL}_2(F)$, and the stabilizer of w_1 in $U(W)$ is the unipotent subgroup of matrices

$$\left\{u(t)\right\} = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in F\right\}.$$

As mentioned before, $U(V) \times U(W)$ embeds into the symplectic group (for the space $V \otimes W$ with alternating form $Tr_{E/F}(\Phi \times \Phi'))$, and embeds into the metaplectic group of $V \otimes W$ for each choice of splitting data $(\psi, \gamma, \chi_1, \chi_2)$.

THEOREM: *Let π be a cuspidal representation of G. Then the following are equivalent:*

(i) *$P(\pi, c, \chi) \neq 0$ for some c and χ.*

(ii) *π has a non-zero theta-lift to H_1.*

(iii) *π is a theta-lift of some cuspidal representation σ of H_1.*

Furthermore, suppose σ is the theta-lift of π to H_1 relative to the specific lifting data $(\psi, \gamma, \chi_1, \chi_2)$; then $P(\pi, c, \gamma^1\chi_2) \neq \{0\}$ if and only if σ has a non-zero Whittaker model $W(\sigma, \psi_c)$ relative to the additive character ψ_c.

Proof: We shall compute the Fourier coefficients of the theta-lift of any φ in π to H_1. For this, we use a Schrödinger model for w_ψ corresponding to the Lagrangian decomposition

$$V \otimes W = (V \otimes w_1) \oplus (V \otimes w_2) .$$

In this model ω_ψ acts on the Schwartz–Bruhat space $\mathcal{S}(V \otimes w_2)$, and we have

$$\omega_\psi(s(g, u(t)))\Phi(v \otimes w_2) = \psi(t\xi^{-1}(v, v))\omega_\psi(s(g, 1))\Phi(v \otimes w_2) .$$

Since $U(V)$ preserves each Lagrangian, we also have

$$\omega_\psi(s(g, 1))\Phi(v \otimes w_2) = \gamma^1(\det\ g)\Phi(vg \otimes w_2) . \tag{2.2.1}$$

Using these formulas, let us compute the c-th Fourier coefficient of a typical element

$$f(h) = \chi_1(\det\ h) \int_{G(F)\backslash G(\mathbb{A})} \chi_2(\det\ g)\Theta_\Phi(\omega_\psi(s(\psi, \gamma)(g, h))\varphi(g)dg$$

in the $(\psi, \gamma, \chi_1, \chi_2)$ lift of π to H_1.

Since the theta-functional in this model is

$$\Theta(\Phi) = \sum_{v \in V(F)} \Phi(v \otimes w_2) ,$$

we have

$$f_{\psi_c}(e) = \int_{F\backslash\mathbf{A}} \overline{\psi(ct)}\left(\int_{G(F)\backslash G(\mathbf{A})} \chi_2(\det g)\varphi(g)\right.$$
$$\left.\times \sum_{v\in V(F)} \omega_\psi(s(\psi,\gamma)(g,u(t))\Phi(v\otimes w_2)dg\right)dt$$

or

$$\int_{G(F)\backslash G(\mathbf{A})} \gamma^1\chi_1(\det g)\varphi(g) \sum_{v\in V(F)} \Phi(vg\otimes w_2)\left(\int_{F\backslash\mathbf{A}} \psi(-ct+t\xi^{-1}(v,v))dt\right) dg.$$

Note that only vectors of length c can contribute to this sum over $V(F)$. So letting v_c be any vector length c, with stabilizer G_c in G, we have

$$\begin{aligned} f_{\psi_c}(e) &= \int_{G(F)\backslash G(\mathbf{A})} \gamma^1\chi_2(\det g)\varphi(g)\left(\sum_{\gamma\in G_c(F)\backslash G(F)} \Phi(v_c\gamma g\otimes w_2)\right) dg \\ &= \int_{G_c(\mathbf{A})\backslash G(\mathbf{A})} \Phi(v_c g\otimes w_2)\left(\int_{G_c(F)\backslash G_c(\mathbf{A})} \gamma^1\chi_1(rg)\varphi(rg)dr\right) dg \\ &= \int_{G_c(\mathbf{A})\backslash G(\mathbf{A})} \gamma^1\chi_2(\det g)\Phi(v_c g\otimes w_2)P(\varphi^g,c,\gamma^1\chi_2)dg \end{aligned} \tag{2.2.2}$$

where $\varphi^g(x) = \varphi(xg)$. Since Φ is an arbitrary Schwartz–Bruhat function on $V(A)$, it is easy to conclude that $f_{\psi_c}(e) = 0$ if and only if $P(\varphi^g, c, \gamma^1\chi_1) = 0$ for all g in $G(\mathbf{A})$. In particular, we conclude $P(\pi, c, \chi) \neq 0$ for some c and χ if and only if π has a non-zero theta-lift to H_1. (Indeed a theta-lift to H_1 is non-zero if and only if some ψ_c-th Fourier coefficient is non-zero.) This proves the equivalence of (i) and (ii) as well as the "*furthermore*" part of the Theorem. As for the equivalence of (ii) and (iii), this has been discussed already in §1.6 of the present paper). ∎

Remarks: (1) In the proof above, it was implicit that c was non-zero. However, a similar computation of the *constant term* of f along U is possible using the same formulas for ω_ψ; the conclusion in this case is that (2.2.2) reduces to the simpler formula

$$f_0(e) = \int_{N(\mathbf{A})\backslash G(\mathbf{A})} \gamma^1\chi_2(\det g)\Phi(v_0 g\otimes w_2)\left(\int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)dr\right) dg$$

since v_0 may be chosen to have stabilizer N in G. (Strictly speaking, this stabilizer is the slightly larger subgroup

$$\left\{\begin{pmatrix} 1 & * & * \\ 0 & \varepsilon & * \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

containing N.) Of course, as φ is assumed cuspidal, this last integral is zero, and hence it follows f must also be cuspidal (as asserted in Proposition 1.5.1).

(2) The relation between the non-vanishing of the lift form $U(V)$ to $U(W)$ and the non-vanishing of certain "periods" is implied in [Wat], although that work ignores the splitting data (ψ, γ) and χ_1, χ_2. ∎

2.3 Periods of stable π

Recall that a cuspidal π is **stable** if and only if *any* theta-series lift of π to *any* unitary group in two variables is zero (cf. Theorem 6.1.1 of [GeRo1]). In particular, any theta-lift of such a π to $H_1 \approx U(1,1)$ is zero. Thus, for stable π,

$$P(\pi, c, \chi) = 0 \quad \text{for all } c \text{ and } \chi \ .$$

This fact is philosophically consistent with Tate's conjecture relating the (existence of) poles for $L(s, \pi \times \xi)$ to the existence of non-trivial periods of π, since $L(s, \pi \times \xi)$ is always entire for stable cuspidal π.

Therefore, we henceforth restrict our discussion of periods to the (more interesting) case of *endoscopic* π.

2.4 Periods of exceptional π

Suppose π is cuspidal but in an A-packet $\prod(\rho)$. *If* the theta-lift of π to $U(1,1)$ were non-trivial, it would automatically be cuspidal, and π itself would then be a theta-lift of this cuspidal σ on $U(1,1)$ (cf. the proof of Proposition 1.6 in Section 1). But by Proposition 6.2.1 of [GeRo1], this would imply $\pi \in \prod(\rho)$ with ρ cuspidal, a contradiction to our assumption that π is exceptional. Thus we conclude (from Theorem 2.2) that the periods of such exceptional π must also all vanish.

2.5 The case of generic endoscopic π

Suppose π is a generic element of a cuspidal endoscopic packet $\prod(\rho)$. In this case, the fact that $L(s,\pi\times\xi)$ has a pole at $s=1$ for some $\xi=\xi_0$ is equivalent to the fact that an appropriate theta-series lift of π to $U(1,1)=H_1$ is non-zero, and the residue of $L(s,\pi\times\xi_0)$ may be expressed directly in terms of the period $P(\pi_0,1,\xi_0)$. (Cf. 5.2.1 of [Ge] and Theorem 8.7 of [GePS].) Using Theorem 2.2, we thus conclude that — for generic cuspidal π — the following are equivalent:

(1) $L(s,\pi\times\xi)$ has a pole at $s=1$ for some fixed $\xi=\xi_0$;

(2') $P(\pi,1,\xi_0)\neq 0$; and

(3') an appropriate theta-series lift of π to $U(1,1)$ is non-zero.

Moreover, suppose the lifting data in (3') is $(\psi,\gamma,\chi_1,\chi_2)$. Then the vanishing (or non-vanishing) of the *compact* periods $P(\pi_0,c,\gamma^1\chi_1)$, with c *not* in $N_{E/F}(E^x)$, can be read off from the vanishing (or non-vanishing) of the ψ_c-th Fourier coefficients of the lifted representation $\sigma=\Theta_{\psi,\gamma}^{\chi_1,\chi_2}(\pi)$ on $U(1,1)$.

Now what is the situation for arbitrary (not necessarily generic) π? By the theory of [GeRo1], conditions (1) and (3') are still equivalent, provided the phrase "to $U(1,1)$" is replaced by the phrase "to some $U(\Phi')$". Thus it remains to analyze the equivalence with (2'), modified to include compact periods as well.

2.6 Compact periods of hypercuspidal π

Let π be any cuspidal endoscopic π on $U(3)$, which we now assume *hypercuspidal* (since the generic case was just treated). In this case, we have already observed that $P(\pi,c,\chi)$ is automatically zero if c is a norm; indeed, for c a norm, $H_c\approx U(1,1)$ and

$$\int_{U(1,1)(F)\backslash U(1,1)(\mathbb{A})}\varphi(h)\chi(\det h)dh=0$$

since the restriction of a hypercuspidal φ to $U(1,1)$ is cuspidal there (and hence orthogonal to all characters).

Thus, we may restrict our attention now to the *compact* periods of a cuspidal π consisting of theta-lifts from a cuspidal σ on some $U(Y)$. In case $U(Y)\approx U(1,1)$, Theorem 2.2 already gives us a criterion for the vanishing of each $P(\pi,c,\chi)$. Thus we further assume now that π lifts from σ cuspidal on an *anisotropic* $U(Y)$.

PROPOSITION 2.6 (A formula for $P(\pi,c,\chi)$): *Suppose $\varphi=\Theta(f,\Phi)$ is the theta-series lift to $U(3)$ of a cusp form f on the anisotropic unitary group $U(Y)$,* *with respect to the lifting data $(\psi, \gamma, \chi_1, \chi_2)$. Then for any χ, and any c not in $N(E^x) \subset F^x$,*

$$P(\varphi, c, \chi) = \alpha(\Phi_2) \int_{U(Y)(F)\backslash U(Y)(\mathbb{A})} \overline{\Theta_{\Phi_1}(1, h)} \chi^*(\det h) f(h) dh.$$

Here $\alpha(\Phi_2)$ is a constant depending on Φ_2, Ev_c denotes the one-dimensional skew-Hermitian space E with $\langle z, w\rangle = c\xi z\bar{w}$, χ^* is the character of $U(Y)$ obtained via the theta-lift of $\chi \circ \det$ on $G_c \approx H_c$ (determined by the same data $(\psi, \gamma, \chi_1, \chi_2)$), and Θ_{Φ_1} is a theta-kernel for the dual pair $(U(Ev_c), U(Y))$ (with more details to be given below in the course of the proof).

Proof: By definition

$$\begin{aligned} P(\varphi, c, \chi) &= \int_{G_c(F)\backslash G_c(\mathbb{A})} \Theta(f, \Phi)(r) \chi(\det r) dr \\ &= \int_{G_c(F)\backslash G_c(\mathbb{A})} \left(\int_{U(Y)(F)\backslash U(Y)(\mathbb{A})} \Theta_{\Phi}^{\psi,\gamma,\chi_1,\chi_2}(r, h) f(h) dh \right) \chi(\det r) dr. \end{aligned}$$

To continue, consider the decomposition

$$V = (v_c) \oplus Z_c$$

where $\langle v_c, v_c\rangle = c\xi$, and Z_c is the orthocomplement of v_c. Then we have the decomposition

$$V \otimes Y = (v_c) \otimes Y \oplus Z_c \otimes Y \tag{2.6.1}$$

of the symplectic space $V \otimes Y$. Note that $(v_c) \otimes Y$ is the symplectic space corresponding to the dual pair $(U(Ev_c), U(Y))$, whereas $Z_c \otimes Y$ corresponds to the pair $(U(Z_c), U(Y))$, with $U(Z_c) \approx H_c$. To compute $P(\varphi, c, \chi)$, we shall realize ω_ψ through its Schrödinger model in $\mathcal{S}(X)$ relative to a suitable polarization

$$V \otimes Y = X \oplus X^* .$$

More precisely, let us assume that this polarization is adapted to the decomposition (2.6.1) in the sense that $X = X_1 \oplus X_2$ with $X_1 = X \cap ((v_c) \otimes Y)$ and $X_2 = X \cap (Z_c \otimes Y)$. Then if Φ in $\mathcal{S}(X)$ is of the form

$$\Phi = \Phi_1 \times \Phi_2 , \quad \text{with} \quad \Phi_j \in \mathcal{S}(X_j) ,$$

we have — for r in the stabilizer G_c of v_c,

$$\Theta_\Phi(r,h) = \Theta^{s_1}_{\Phi_1}(1,h)\Theta^{s_2}_{\Phi_2}(r,h) .$$

Here $\Theta^{s_1}_{\Phi_1}$ and $\Theta^{s_2}_{\Phi_2}$ are the theta-kernels for the dual pairs $(U(Ev_c), U(Y))$ and $(H_c, U(Y))$ respectively, and s_1 and s_2 denote the appropriate embeddings inherited from $s = s(\psi, \gamma, \chi_1, \chi_2)$. In particular, we can write

$$\begin{aligned} P(\varphi, c, \chi) &= \int_{G_c(F)\backslash G_c(\mathbb{A})} \int_{U(Y)(F)\backslash U(Y)(\mathbb{A})} \overline{\Theta_{\Phi_1}(1,h)\Theta_{\Phi_2}(r,h)} f(h)dh\ \chi(\det r)dr \\ &= \int_{U(Y)(F)\backslash U(Y)(\mathbb{A})} \Theta_2(\chi)(h)\overline{\Theta_{\Phi_1}(1,h)} f(h)dh \end{aligned}$$

with $\Theta_2(\chi)$ the theta-lift

$$\int_{G_c(F)\backslash G_c(\mathbb{A})} \overline{\Theta_{\Phi_2}(r,h)}\chi(\det r)dr$$

of χ on $G_c \approx U(Z_c) \approx H_c$ to $U(Y)$.

To complete the proof, it remains to prove that $\Theta_2(\chi)$ is itself (a multiple $\alpha(\Phi_2)$ of) some character χ^* on $U(Y)(\mathbb{A})$. For this, we may assume $\Theta_2(\chi)(h) \not\equiv 0$ (since otherwise $P(\varphi, c, \chi)$ is automatically zero, and we don't need a formula for it!) By Part (a) of Proposition 1.2, we know at least that the functions in $\Theta_2(\chi)$ generate a sum of irreducible representations $\sigma(\chi)$ of $U(Y)$. (In the proof of Proposition 1.2, "cuspidality" of $\Theta_2(\chi)$ was assumed only to ensure that the lifted automorphic representation decomposed *directly*; since Y is anisotropic here, the compactness of $U(Y)(F)\backslash U(Y)(\mathbb{A})$ makes this assumption redundant.) Moreover, almost every local component of $\sigma(\chi)$ is the Howe lift of χ. Thus it will suffice to prove that at least one of these local lifts again generates a character.

In fact, let v be any place where both $U(Y)_v$ and $U(Z_c)_v$ are isomorphic to $GL_2(F_v)$ (this happens for infinitely many v). Then the Howe lift (for the *type II* pair (GL_2, GL_2)) can be constructed explicitly using Tate–Godement–Jacquet integrals as on p.65 of [MVW]. In this case, modulo possible twisting by unitary characters of the determinant, ω_ψ restricted to $GL_2 \times GL_2$ acts in $\mathcal{S}(M(2,2))$ via the natural actions of GL_2 on the right and left of 2×2 matrix space $M(2,2)$; cf. [MVW] p.62. Moreover, the integrals

$$\int_{GL_2(F_v)} \Phi(xg)\chi_v(\det x)d^x x ,$$

or possibly their residue, provide elements of $\mathrm{Hom}(\omega_\psi\otimes\chi_v,\chi_v^{-1})$. This establishes that the Howe lift of χ to $U(Y)$ is again a character, and completes the proof of Proposition 3.1. ∎

2.7 Examples of π with $P(\pi,c,\chi)=0$ for all c and χ

Let us call a cuspidal representation σ of $U(Y)$ **theta-stable** if *any* theta-lift of σ to any unitary group $U(1)=U(Ev_c)$ is zero. In classical language, this means that σ is *not* a theta-series constructed from a (Hermitian) form in one variable. Clearly, "most" cuspidal σ on $U(Y)$ are theta-stable (we return to the meaning of theta-stability in §3.1 below).

PROPOSITION: *There exist (hypercuspidal) endoscopic cuspidal π on $U(3)$ with the property*

$$P(\pi,c,\chi)=0 \quad \text{for all } c \text{ and } \chi\ .$$

Indeed, if σ is a theta-stable cuspidal representation of an anisotropic $U(Y)$ in two variables, and the lifting data $(\psi,\gamma,\chi_1,\chi_2)$ is chosen so that the lift of σ to $U(3)$ is non-zero, then (any irreducible component π of) $\pi=\Theta_{\psi,\gamma}^{\chi_1\chi_2}(\sigma)$ is a cuspidal such π on $U(3)$.

N.B: By Proposition 1.8 we know it is always possible to find $(\psi,\gamma,\chi_1,\chi_2)$ as in the Proposition above. By Theorem 2.2, we also know that such π as constructed above must always lift to zero on $U(1,1)$. ∎

Proof of the Proposition: Because σ is *theta*-stable, Proposition 1.5.2 implies that its theta-lifts to $U(3)$ are cuspidal; moreover, since $U(Y)$ is compact, these lifts are also hypercuspidal (cf. the Concluding Remark of §1.7). Therefore, as observed at the beginning of this Section, the *non-compact* periods $P(\pi,c,\chi)$ must automatically vanish.

So suppose now that c is *not* a norm from E, and χ is any character of $E^1\backslash E^1(\mathbf{A})$. Then by the formula of Proposition 2.6,

$$P(\varphi,c,\chi)=\alpha(\varphi_2)\int_{U(Y)(F)\backslash U(Y)(\mathbb{A})}\overline{\Theta_{\Phi_1}(1,h)}\chi^*(h)f(h)dh\ , \tag{2.7.1}$$

with $\Theta_{\Phi_1}(\cdot,h)$ a theta-kernel for the pair $(U(Ev_c),U(Y))$ (with respect to certain lifting data s). But

$$\chi^*(\det h)\Theta_{\Phi_1}(\cdot,h)$$

is clearly another such theta-kernel (this time with respect to the embedding $s^*(\cdot, h) = \chi^*(\det h)s(\cdot, h)$). Thus the right-hand side of (2.7.1) is just a theta-lift of f to $U(Ev_c)$, and the fact that $\sigma = \{f\}$ is (assumed) theta-stable implies that $P(\pi, c, \chi)$ must vanish. ∎

2.8 Examples of hypercuspidal π with not all $P(\pi, c, \chi)$ zero

PROPOSITION: *There exist hypercuspidal endoscopic π such that*

$$P(\pi, c, \chi) \neq 0 \quad \textit{for some } c \textit{ and } \chi \ ,$$

namely: Take a cuspidal σ on $U(W) \approx U(1,1)$, a character ψ of $\mathbb{A}/F$ such that $\mathcal{W}(\sigma, \psi) = \{0\}$, and lifting data $(\psi, \gamma, \chi_1, \chi_2)$ such that $\Theta_{\psi,\gamma}^{\chi_1\chi_2}(\sigma) \neq \{0\}$ on $U(3)$; then each irreducible component of $\Theta_{\psi,\gamma}(\sigma)$ is an irreducible hypercuspidal endoscopic π of the above type.

Example: Let σ on $U(W)(\mathbb{A})$ be the theta-series lift of a character χ on $U(Ev_1)$ for some lifting data $(\psi', \gamma, \chi_1, \chi_2)$. In this case, σ will be non-zero (regardless of the data $(\psi, \gamma, \chi_1, \chi_2)$, just as theta-lifts from $U(Ev_1)$ to $U(3)$ must always be non-zero; cf. §3 of [Ge]). Moreover, it is easy to check that $\mathcal{W}(\sigma, \psi) = \{0\}$ if ψ is *not* in the orbit of ψ', i.e., $\psi(x) \neq \psi'(ax)$ for *any norm a* from E^x. Thus such a σ (and ψ) provide examples of the type required above. (These σ, of course, are just the $U(1,1)$ analogues of the dihedral cusp forms on GL(2) constructed via theta-series by Hecke.) ∎

Proof of the Proposition: Taking σ and ψ as in the statement of the Proposition, we use Proposition 1.8 to find lifting data $(\psi, \gamma, \chi_1, \chi_2)$ so that $\Theta_{\psi,\gamma}^{\chi_1,\chi_2}(\sigma) \neq \{0\}$. Note (by Proposition 1.7) that $\pi = \Theta_{\psi,\gamma}(\sigma)$ can have no Whittaker model; moreover, it will be endoscopic as soon as it is cuspidal (Proposition 6.2.1 of [GeRo1]). Therefore, it remains only to show that π is cuspidal (since a cuspidal π is hypercuspidal as soon as its Whittaker models vanish).

We begin by showing directly that each φ in $\Theta_{\psi,\gamma}(\sigma)$ is hypercuspidal. To this end, we compute the constant term of φ along U exactly as we computed the non-constant Fourier coefficients in the proof of Proposition 1.8. The end result, analogous to (1.5.4), is the formula (valid for the full lift $\Theta(\sigma)$, hence also for P in any summand):

$$\varphi_U(e) = \Phi_1(0) \int_{U(W)(F)\backslash U(W)(\mathbb{A})} \gamma^1 \chi_1(\det h) \Theta_{\Phi_2}(1, h) f(h) dh \tag{2.8.1}$$

$$+\int_{U_H(\mathbf{A})\backslash U(W)(\mathbf{A})}\gamma^1\chi_1(\det h)\Phi_1(hw_{-1})\left(\int_{U_H(F)\backslash U_H(\mathbf{A})}\Theta_{\Phi_2}(1,uh)f(uh)du\right)dh,$$

where w_{-1} is the isotropic element of W whose stabilizer in $U(W)$ is just the unipotent group

$$U_H \approx \{\begin{pmatrix}1 & s\xi\\ 0 & 1\end{pmatrix}\}$$

(and the first integral is the contribution from the zero vector in $\{w : (w,w) = 0\}$). To prove that π is hypercuspidal we need to prove that $\varphi_U \equiv 0$.

To see that the *second* term describing φ_U vanishes, consider the Fourier expansions of $\Theta_{\Phi_2}(1,uh)$ and $f(uh)$ along the compact abelian group $U_H(F)\backslash U_H(\mathbf{A}) \approx F\backslash\mathbf{A}$. As mentioned in the Example above, it is easy to check that the ψ'-th Fourier coefficients of $\Theta_{\Phi_2}(1,uh)$ are zero unless ψ' is in the orbit of ψ^{-1}. On the other hand, the ψ-th Fourier coefficients of f in σ are *assumed* to vanish; thus the inner integral in the second term above clearly vanishes.

N.B.: This argument is completely different from the one used in §1.5 to analyze the contribution of the "second term" describing φ_U. Here we conclude that this term is zero by assuming that $\mathcal{W}(\sigma,\psi) = \{0\}$ (in addition to σ being cuspidal); there we only assumed σ was cuspidal, but needed to integrate over $U(\mathbf{A})N(F)\backslash N(\mathbf{A})$ before getting zero. ∎

To see that the *first* term in (2.8.1) also vanishes, note that this expression is just ($\Phi_2(0)$ times) the theta-series lifting of $\gamma^1(\det h)f(h)$ to $U(E)$ (with respect to the data $(\psi, g, \chi_1, \chi_2)$). But by the Lemma stated and proved below, this theta-lifting must be zero, and hence $\varphi_U \equiv 0$.

To complete the proof (assuming the Lemma below), we use Proposition 6.2.1 of [GeRo1] knowing now that $\Theta(\sigma)$ is hypercuspidal — hence cuspidal) to conclude that each irreducible component of $\Theta_{\psi,\gamma}(\sigma)$ generates an irreducible hypercuspidal endoscopic representation π of $U(3)$. Moreover,since π itself is a theta-lift from σ cuspidal on $U(1,1)$, it must have a non-zero lift back to $U(1,1)$ (cf. Part (b) of Proposition 1.2), and hence by Theorem 2.2, some $P(\pi, c, \chi)$ is non-zero. ∎

It remains to prove:

LEMMA 2.8.2: *Suppose $\mathcal{W}(\sigma,\psi) = \{0\}$, and fix any data (γ,χ_1,χ_2) for the lifting between $U(W)$ and $U(E)$. Then the $(\psi,\gamma,\chi_1,\chi_2)$ lifting of σ to $U(E)$ is zero.*

Proof: Suppose not. Then the lift to $U(E)$ is non-zero, hence irreducible, and generates some character χ. In particular, σ_v is then the Howe lift of χ_v for each v which does not split in E, and by a familiar argument, this implies $\mathcal{W}(\sigma_v, \psi'_v) = 0$ unless ψ'_v is in the orbit of ψ_v, i.e., of the form $\psi_v(b_v x)$ for some norm b_v from E_v^x. So choose a global b in F^x which is *not* a norm from E^x, and such that $\mathcal{W}(\sigma, \psi_b) \neq \{0\}$. (This is always possible since $\mathcal{W}(\sigma, \psi') \neq \{0\}$ for some ψ', and by hypothesis $\mathcal{W}(\sigma, \psi_a) = \{0\}$ if a *is* a norm from E^x.) If v is a finite place such that v does *not* split in E, *and* such that b is not a norm from E_v^x, then $\mathcal{W}(\sigma_v, (\psi_b)_v) \neq \{0\}$, contradicting the fact that b_v must be a norm. ∎

Remark: The construction outlined above must yield *all* hypercuspidal endoscopic π with the property that $P(\pi, c, \chi) \neq 0$ for some c and χ. To see this, one can argue "backwards" from the identity (2.8.1). ∎

2.9 Existence of algebraic cycles not spanned by modular curves

We shall prove that there exist cycle classes on a Picard modular surface S_K not spanned by the modular curves embedded in S_K. We start by recalling some definitions and notation; for further background, see [Mo].

Let $E/\mathbb{Q}$ be a quadratic *imaginary* extension, and G the quasi-split unitary group $U(3)$ relative to $E/\mathbb{Q}$. Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup, and set $\mathfrak{X} = G(\mathbb{R})/K_\infty$. Then $\mathfrak{X}$ is a Hermitian symmetric domain, isomorphic to the complex two-ball. For each open compact subgroup $K \subset G(\mathbb{A}_f)$, let

$$S_K = G(\mathbb{Q})\backslash \mathfrak{X} \times G(\mathbb{A}_f)/K$$

be the corresponding Shimura variety. For K sufficiently small (which we henceforth assume), S_K can be embedded in projective space as the set of complex points of a smooth quasi-projective variety S_K of dimension 2. Its Baily–Borel compactification is denoted by $S_K^{\#}$, and its canonical smooth compactification by S_K^*. Moreover, there is a surjective map

$$\beta : H^2(S_K^*, \mathbb{C}) \longrightarrow IH^2(\mathbb{C}) = IH^2(S_K^{\#}, \mathbb{C}) ,$$

where IH^2 denotes the intersection cohomology of $S_K^{\#}$ (cf. [BlRo]).

Now denote by $\mathcal{Z}$ the image under β of the subspace of $H^2(S_K^*, \mathbb{C})$ generated by the classes of *algebraic cycles* of codimension one on S_K^*. Examples of such classes may be constructed as follows. For each ξ in $\mathbb{Q}^+$, let H_ξ denote the corresponding unitary group in two variables introduced in §2.1. Recall that we can fix an embedding of H_ξ into G which is unique up to $G(\mathbb{Q})$-conjugacy. The symmetric space attached to H_ξ is the upper half-plane $\mathfrak{h}$, and for each open compact subgroup $K_\xi \subset H_\xi(\mathbf{A}_f)$,

$$C_{K_\xi} = H_\xi(\mathbb{Q})\backslash\mathfrak{h} \times H_\xi(\mathbf{A}_f)/K_\xi$$

is isomorphic to the union of *modular curves*. Finally, if $K_\xi \subset K_f$, we obtain an embedding of C_{K_ξ} into S_K. Then C_{K_ξ} defines a class in $\mathcal{Z}$, and we let M denote the subspace of $\mathcal{Z}$ generated by these fundamental classes under the natural action of the Hecke algebra.

The purpose of this paragraph is to prove the following:

THEOREM 2.9: *For K sufficiently small, M is a proper subspace of $\mathcal{Z}$. In particular, there exist cycle classes which are not in the subspace spanned by the fundamental classes of the modular curves embedded in S_K.*

The idea of the proof is to reduce the theorem to an assertion about theta-series liftings between $U(2)$ and $U(3)$. However, to even state this assertion, we need to recall some (irreducible) representations and results related to the decomposition of the cohomology $IH^2(\mathbb{C})$.

First, let $\Pi^0_\infty = \{\pi^+, \pi^-, \pi_0\}$ denote the L-packet consisting of the three discrete series representations π such that $H^*(\mathrm{Lie}(G(\mathbb{R})),\ K_\infty, \pi) \neq 0$. We recall that this Lie algebra cohomology has a Hodge decomposition, and we may choose the labelling so that the Hodge type of π^0 is (1,1). The cohomology $IH^2(\mathbb{C})$ can then be decomposed into a direct sum

$$IH^2(\mathbb{C}) = \bigoplus_{\pi_f} H(\pi_f)\ ,$$

where each admissible representation π_f of $G(\mathbf{A}_f)$ is such that $\pi_\infty \otimes \pi_f$ is cuspidal for some $\pi_\infty \in \Pi^0_\infty$, and each direct summand $H(\pi_f)$ is an isotypic component under the action of the Hecke algebra. If $\mathrm{Inf}(\pi_f)$ denotes the set of π_∞ in Π_∞ such that $\pi_\infty \otimes \pi_f$ occurs in the cuspidal spectrum, then as a module over the K_f-Hecke algebra,

$$H(\pi_f) = V(\pi_f) \otimes \pi_f^{K_f}\ ,$$

where $\pi_f^{K_f}$ is the space of K_f invariants, and $V(\pi_f)$ is a vector space of dimension

$$d(\pi_f) = \mathrm{Card}(\mathrm{Inf}(\pi_f))\ .$$

PROPOSITION 2.9.1 (cf. [BlRo]): *Suppose that* $\mathrm{Inf}(\pi_f) = \{\pi^0\}$. *Then* $H(\pi_f)$ *is a subspace of* $\mathcal{Z}$.

In view of this result, Theorem 2.9 reduces to the following:

PROPOSITION 2.9.2: *There exists a cuspidal endoscopic L-packet* Π *with* $\Pi_\infty = \Pi^0_\infty$ *such that for some* π_f *in* Π_f, *the following hold:*

(a) $\mathrm{Inf}(\pi_f) = \{\pi^0\}$; *and*

(b) $\pi = \pi^0 \otimes \pi_f$ *is not a theta-lift from* $U(1,1)$.

Indeed, condition (b) implies (by our Theorem 2.2) that *all* the periods of π are zero. On the other hand, condition (a) implies that $H(\pi_f)$ is a subspace of $\mathcal{Z}$. It follows that $\mathcal{Z}$ is not spanned by the classes associated to the cycles C_{K_s}. (For an interpretation of period integrals in terms of the intersection pairing on $IH^2(\mathbb{C})$, see [HLR].)

To prove the Proposition, we fix once and for all a global splitting of the metaplectic group over $G \times U(1,1)$ corresponding to the datum $(\psi, \gamma, \chi_1, \chi_2)$. The corresponding theta lift of a cuspidal representation σ of $U(1,1)$ is simply denoted $\Theta(\sigma, \psi)$ (where ψ is the non-trivial additive character of $\mathbb{A}/F$ appearing in the datum $(\psi, \gamma, \chi_1, \chi_2)$).

Now let $\rho = \rho_2 \times \rho_1$ be any cuspidal L-packet on $H = U(1,1) \times U(1)$ such that

(i) $\Pi(\rho_\infty) = \Pi^0_\infty$ and

(ii) $\rho_{1\infty}$ is trivial.

(There exists a unique discrete series packet ρ_∞ satisfying (i) and (ii); cf. §3 of [Ro2].) For distinct *odd* primes p and q which remain prime in E, and $v = p$ or q, let ρ^0_v be an L-packet on $H(\mathbb{Q}_v)$ which consists of two supercuspidal representations. If we set $\Pi^0_v = \Pi(\rho^0_v)$, then Π^0_v consists of four super cuspidal representations of $G(\mathbb{Q}_v)$ (cf. 12.2 of [Ro1]). Moreover, a standard application of the trace formula shows that we may choose a cuspidal L-packet ρ as above so that ρ is stable and $\rho_v = \rho^0_v$ for $v = p, q$.

Finally, setting $\Pi = \Pi(\rho)$, and fixing data $(\psi, \gamma, \chi_1, \chi_2)$, we let Σ denote the unique cuspidal L-packet on $U(1,1)$ such that a *non-zero* irreducible component of the corresponding theta lift of σ to $U(3)$ belongs to Π if and only if $\sigma \in \Sigma$ (cf. Proposition 6.2.1 of [GeRo1]). Note that Howe's conjecture is already proved for the representations σ_v since $v = p$ or q is assumed odd. Thus we can introduce "the" local Howe lifts $\Theta(\sigma_v, \psi_v)$ for each σ_v in Σ_v and ψ_v a local non-trivial additive character. If we let X_v denote the set of these representations, then X_v

is a subset of Π_v, and we claim that Proposition 2.9.2 follows from the assertion

$$(*) \qquad \mathrm{Card}(X_v) \leq 2 .$$

Indeed, $(*)$ implies we may choose π_p^0 to be an element of Π_p^0 which does <u>not</u> belong to X, and set

$$\pi_f = \pi_p^0 \otimes \pi_q \otimes \pi'_f ,$$

where π'_f is the tensor product of components π_w with $w \neq p, q, \infty$. But for any π'_f, there exists a choice of π_q in Π_q such that $\mathrm{Inf}(\pi_f) = \pi^0$ (cf. Proposition 13.1.2(b) and Theorem 13.3.7 of [Rol]). So with this choice of π_q, the representation π_f satisfies the conditions of Proposition 2.9.2, and it remains only to prove $(*)$.

Recall first that the elements of Σ_v are all of the form σ_v^a for some a in F_v^x, where the exponent a denotes conjugation by the matrix

$$\delta_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} .$$

(Cf. Chapter 11 of [Rol].) Moreover, the isomorphism class of σ_v^a depends only on a modulo $N_{E/F}(E_v^*)$, and the set X consists of the theta-lifts $\Theta(\sigma_v^a, \psi_v^b)$ where *a and b* range through F^x modulo $N_{E/F}(E_v^x)$. (Cf. [GeRol], §3). Hence $\mathrm{Card}(X) \leq 4$. To prove that at most two of these elements are distinct, it suffices to prove that $\Theta(\sigma_v^a, \psi_v^b)$ is isomorphic to $\Theta(\sigma_v, \psi_v^{ab})$.

We show more generally that $\omega_{\psi^a}(s(\psi^a,\gamma)(h,g))$ is equivalent to $\omega_\psi(s(\psi,\gamma)(h^a,g))$. For this, define a character of order two of the norm one elements E^1 by the formula

$$\chi_a\left(\frac{y}{\bar{y}}\right) = (a, y\bar{y}).$$

It follows [GeRol] (p.458 and 3.1.2) that

$$s(\psi_a,\gamma)(h,g) = \chi_a(\det(h\otimes g))s(\psi,\gamma)(h,g), \quad \text{and} \quad \det(h\otimes g) = \det(h)^3\det(g)^2.$$

This implies

$$s(\psi^a,\gamma)(h,g) = \chi_q(\det\ h)s(\psi,\gamma)(h,g) ,$$

and hence

$$(**) \qquad \omega_{\psi^a}(s(\psi^a,\gamma)(h,g)) = \chi_a(\det\ h)\omega_{\psi^a}(s(\psi,\gamma)(h,g)) .$$

On the other hand, there exists an automorphism $\tilde{\sigma}$ of $\mathrm{Mp}(V \otimes W)$ which lifts the adjoint action $\mathrm{ad}(\delta_a)$ of δ_a on $\mathrm{Sp}(V \otimes W)$ ([MVW], p.36). Hence

$$\tilde{\delta}(s(h,g)) = \chi(h,g)(s(h^a,g)$$

for some character $\chi(h,g)$ of $H \times G$. But $\omega_\psi(\tilde{\delta}(x))$ is equivalent to $\omega_{\psi^a}(x)$, and therefore

$$\chi(h,g)\omega_\psi(s(\psi,\gamma)(h^a,g))$$

is equivalent to

$$\omega_{\psi^a}(s(\psi,\gamma)(h,g)) = \chi_a(\det(h \otimes h))\omega_{\psi^a}(s(\psi^a,\gamma)(h,g)) \,.$$

Since (**) implies that this latter expression equals

$$\chi_a(\det(h))\omega_{\psi^a}(s(\psi^a,\gamma)(h,g)),$$

it remains to check that $\chi(h,g) = \chi_a(\det(h \otimes g))$ for (h,g) in the diagonal subgroup of $U(V) \times U(W)$ (since these elements give all possible determinants in E^1). Observe that $\mathrm{ad}(\delta)$ fixes the image in $\mathrm{Sp}(V \otimes W)$ of the product of $U(V)$ and $U(W)$. On the other hand, formula (3.1.3) of [GeRol] describes how the action of such elements (and more generally elements preserving the relevant polarization) depend on Weil's constant γ and hence on ψ. From this it follows that $\chi = \chi_a$.

3. Concluding Remarks and Open Problems

3.1 Stability of cusp forms on $U(2)$ (concluded)

For each automorphic character $\theta = \theta_1 \times \theta_2$ of $U(\Phi')$, there is an (endoscopic) L-packet $\rho(\theta)$ on $U(\Phi')(\mathbf{A})$ corresponding to the functorial lift of θ through the L-group morphism

$$\rho: \ {}^LU(1) \times U(1) \longrightarrow {}^LU(\Phi')$$

(cf. Chapter 4 of [Rol]). Following [Rol], we call a cuspidal automorphic representation σ of $U(\Phi')(\mathbf{A})$ **stable** if it is not a member of any such packet $\rho(\theta)$.

On the other hand, in §2 we encountered a notion of stability for σ based on the theory of theta-series liftings, i.e. σ is *stable* (or "*theta-stable*") if for *any* lifting data $(\psi,\gamma,\chi_1,\chi_2)$, the theta-lift of σ to $U(E)$ is zero.

FACT 3.1.1: *A representation σ is theta-stable if and only if it is stable in the sense of [Ro1].*

This assertion is entirely analogous to Theorem 6.1.1 of [GeRo1] which implies that a cuspidal π on $U(3)$ is stable (in the sense of [Ro1]) if and only if its theta-lifts to any $U(\Phi')$ are always zero. Moreover, the proofs are similar: the "if" direction is again the harder one, and it can be proved using a Shimura-type integral (this time for $L(s, \sigma \times \xi)$ in place of $L(s, \pi \times \xi)$). We omit further details, except to say that for $U(1,1)$ the required Shimura-type integral is very similar to that used for the symmetric square L-function in [Ge Ja].

Now let us fix the dual pair $(U(\Phi'), U(3))$, where the *skew* form on E is $\xi x\bar{w}$. For fixed lifting data

$$s = s(\psi, \gamma, \chi_1, \chi_2) ,$$

let us call a cuspidal representation σ of $U(\Phi')(\mathbf{A})$ s-**theta stable** (or just s-**stable**) if the s-theta lift of $\sigma \otimes \gamma^1$ to $U(E)$ is zero. Note that

(1) we are not dealing with the lift of σ to $U(E)$, but rather σ *twisted by* $\gamma^1 \circ \det$ (where γ^1, as usual, denotes the restriction of γ to norm one elements);

(2) σ is *theta-stable* if and only if it is s-stable for all possible s.

3.2 Basic open problems

PROBLEM A: *Given π in $\prod(\rho)$, a two variable unitary group H_c, and fixed lifting data $(\psi, \gamma, \chi_1, \chi_2)$, what conditions on π ensure that π has a non-zero theta-lift to H_c (for this fixed lifting data $\psi, \gamma, \chi_1, \chi_2$)?*

As Section 2 shows, we cannot expect the answer to be in terms of L and ε conditions, since the non-vanishing of $\Theta(\pi)$ clearly depends on *where* π lies in the L-packet. On the other hand, we do have one acceptable answer for H_1, namely that $\Theta(\pi)$ will be non-zero on H_1 (for some lifting data) if and only if $P(\pi, c, \chi) \neq 0$ for some c and χ. A nicer answer would be the following:

CONJECTURE: *π has a non-zero theta lift to H_c (with respect to the data $\psi, \gamma, \chi_1, \chi_2$) if and only if each component π_v is a Howe lift from $(H_c)_v$ (with respect to the corresponding local lifting data).*

In the reverse direction, we have a similar problem:

Problem B: *Given a cuspidal σ on H_c, and lifting data $(\psi, \gamma, \chi_1, \chi_2)$, when is the theta lift of σ to $U(3)$ non-zero?*

As already remarked, our result that such a lift is always non-zero for *some* suitably chosen lifting data has the obvious shortcoming that we can't easily specify *which* data works (and hence we are also unable to specify to which L-packet $\prod(\rho)$ this lift belongs.

Problem C: *Given a cuspidal packet $\rho = \rho_1 \times \rho_2$ of $U(1,1) \times U(1)$, can we describe the set*

$$\{\pi \in \prod(\rho) \colon \pi \ \text{ cuspidal}\}$$

explicitly as the set of Θ-lifts

$$\{\Theta_{\psi,\gamma}^{\chi_1\chi_2}(\sigma)\}_{\psi,\sigma}$$

where ψ varies through the class of additive characters of $\mathbb{A}/F$, (γ, χ_1, χ_2) is some lifting data specified completely by ρ (and the μ defining $\xi_H^\mu : {}^LH \longrightarrow {}^LG$), and σ exhausts a given set of cuspidal representations of certain unitary groups in two variables?

Note that Theorem 6.1 of [Ge] (and especially the equality 6.1.3) gives a precise solution to this question in case ρ is one-dimensional, i.e. for the A-packet $\prod(\rho)$ in place of the cuspidal L-packet $\prod(\rho)$. Theorem 6.4 of [Ge] falls short of this goal for the latter L-packet $\prod(\rho)$ because we are imprecise there about the lifting data and unitary groups involved.

Problem D (Local Analogue of C): *For a given L-packet ρ of $U(1,1) \times U(1)$, can we describe the finite set*

$$\{\pi : \pi \in \prod(\rho)\}$$

as an explicit collection of Howe lifts from $U(1,1)$ or the unique compact $U(\Phi')$?

We expect that — aside from a few exceptional (specifiable) ρ — each π in $\prod(\rho)$ is a Howe lift from *either* $U(1,1)$ *or* the compact $U(\Phi')$, but *not both*; in this case, it should be possible to specify σ and lifting data (γ, χ_1, χ_2) in terms of ρ, and then prove that $\prod(\rho)$ is the collection of Howe lifts $\mathrm{Howe}_{\psi,\gamma}^{\chi_1\chi_2}(\sigma^*)$, as ψ varies through classes of ψ, and σ^* equals σ or its Jacquet–Langlands lift to the compact form of $U(1,1)$.

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